Exercise 25

Show that the solution (2) has initial data at time $t_0 = \frac{3L}{2c}$ given by

$$u(x, t_0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(x, t_0) = \frac{\pi c}{L} \sin \frac{\pi x}{L}.$$

Solution

Equation (2) is a solution to the wave equation on a finite interval with fixed ends.

$$u(x, t) = \sin \frac{\pi x}{L} \cos \frac{\pi c t}{L} \tag{2}$$

Differentiate it with respect to t.

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\left(\sin \frac{\pi x}{L} \cos \frac{\pi c t}{L}\right) \\ &= -\frac{\pi c}{L} \sin \frac{\pi x}{L} \sin \frac{\pi c t}{L}\end{aligned}$$

Now plug in $t = \frac{3L}{2c}$ to these formulas for $u(x, t)$ and $\partial u/\partial t$.

$$\begin{aligned}u\left(x, \frac{3L}{2c}\right) &= \sin \frac{\pi x}{L} \cos \left[\frac{\pi c}{L}\left(\frac{3L}{2c}\right)\right] \\ &= \sin \frac{\pi x}{L} \cos \frac{3\pi}{2} \\ &= 0 \\ \frac{\partial u}{\partial t}\left(x, \frac{3L}{2c}\right) &= -\frac{\pi c}{L} \sin \frac{\pi x}{L} \left[\sin \frac{\pi c}{L}\left(\frac{3L}{2c}\right)\right] \\ &= -\frac{\pi c}{L} \sin \frac{\pi x}{L} \sin \frac{3\pi}{2} \\ &= \frac{\pi c}{L} \sin \frac{\pi x}{L}\end{aligned}$$